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A Refineable Space of Smooth Spline Surfaces of Arbitrary Topological Genus

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It is shown that parametrical smoothness conditions are sufficient for modeling smooth spline surfaces of arbitrary topology if degenerate surface segments are accepted. In general, degeneracy, i.e., vanishing partial derivatives at extraordinary points, is leading to surfaces with geometrical singularities. However, if the partial derivatives of higher order satisfy certain conditions, the existence of a regular smooth reparametrization can be guaranteed. So, degeneracy is no fundamental obstacle to generating surfaces which are smooth in the sense of differential geometry. Besides its striking simplicity the approach presented here admits the construction of smooth spline spaces which have a natural refinement property. Thus, various algorithms based on subdivision of tensor product B-spline surfaces become available for surfaces of general type.

INTRODUCTION

The *spline space* $\mathscr{S}_\Omega := \{\mathbf{x}: \Omega \mapsto \mathbb{R}^3\}$ is a class of functions over some domain $\Omega = \{\omega_i \subset \mathbb{R}^2, i \in I \subseteq \mathbb{N}\}$ formed by a set of compact *subdomains* ω_i provided with a *connectivity relation* $\mathscr{C}$. The topological structure of the domain Ω can be visualized conveniently by a two-dimensional mesh as indicated in Fig. 3.1. The elements of $\mathscr{S}_\Omega$ are called *spline surfaces over* Ω. The restrictions $\mathbf{x}^i$ of $\mathbf{x} \in \mathscr{S}_\Omega$ to ω_i are called *patches* or *segments* and the connectivity relation describes how to link them. Points on the graph of a spline function which are uniquely assigned to one segment are called *interior points*, points which are common to exactly two segments form *edges*, and points shared by $n > 2$ segments are called *vertices of order n*.

Here, the segments are assumed to be quadrilateral analytic patches and so the domain Ω is *uniform* in the sense that it consists of copies of

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only one set, namely the unit square $\omega := [0, 1]^2$; hence, $\Omega = \omega \times I$. For reasons discussed in Section 1, vertices of order n are called *regular* if $n = 4$ and *extraordinary* otherwise. If all vertices are regular, $\mathscr{S}_\Omega$ is called *regular*. However, as a consequence of Euler's theorem relating the number of faces, edges, and vertices of polyhedra, regular spline spaces only admit the modeling of surfaces which are homeomorphic to a torus or a plane or parts of it (see [10]). This well-known observation justifies the necessity of *irregular* spline spaces including extraordinary vertices. An irregular spline space is called *semiregular* if the extraordinary vertices are separated, i.e., if there is no edge connecting any two of them. Figure 3.1 shows a part of a semiregular domain with two extraordinary vertices of orders three and five, which are separated by a regular vertex of order four.

When modeling smooth spline surfaces, i.e., surfaces with a well-defined and continuously varying normal vector, appropriate smoothness conditions for connecting adjacent patches must be specified. *Parametrical smoothness*, i.e., equality up to the sign of the transversal partial derivatives along common boundary curves, is a natural and convenient choice in the regular case. However, it is commonly considered to be too restrictive for edges incorporating an irregular vertex. So, the concept of *geometrical smoothness* has been developed [1, 2, 6–10, 14, 19] for solving the problem. Besides requiring polynomials of high degree, the major drawback of this approach is the absence of a natural refinement property as provided by subdivision algorithms for tensor product spline surfaces.

In this report we present a different way of constructing smooth spline surfaces of arbitrary topological type. The idea is to use exclusively parametric smoothness conditions and to overcome the related difficulties by introducing surface patches which are *degenerate* in the sense that their partial derivatives vanish at extraordinary vertices. This method was proposed in [15–17] and verified strictly in [18, 21] for polynomial patches. Here it is shown that the method can be generalized to degenerate patches based on arbitrary analytic basis functions, thus extending the range of applicability to most spline models currently in use. Besides its striking simplicity the major benefit of this approach is that it admits the construction of spline spaces which have a natural refinement property. This should facilitate the use of many powerful tools and algorithms based on subdivision of tensor product B-splines to spline surfaces of arbitrary topology. To enumerate only a few of them, think of surface rendering and determining cross sections of spline surfaces in computer-aided design or the recently developed concepts of wavelets and hierarchical basis in the field of approximation theory.

The paper is organized as follows. In the first section, sufficient regularity conditions for degenerate analytic surface patches are derived. Roughly speaking, a surface segment is degenerate if the partial derivatives of order

(1, 0), (0, 1), and (1, 1) are zero at one of its corners, and in general, this will cause a cusp-like geometrical singularity. However, regularity, i.e., the existence of a regular smooth reparametrization, can be guaranteed if the partial derivatives of order (2, 0), (2, 1), (1, 2), and (0, 2) are coplanar and properly arranged at this point. Surface segments of this type are called *D-patches*. Further, the behavior of the main curvatures near the singular point is studied. In the second section, the conditions for D-patches are applied to polynomial patches in Bernstein–Bézier form. It turns out that degeneracy is equivalent to four coalescing Bézier points while regularity requires certain neighboring Bézier points to be coplanar. In the third section, a space $\mathscr{S}_\Omega$ of spline surfaces incorporating D-patches is introduced. It is based on bicubic polynomial patches of first order joining parametrically smoothly. The elements of $\mathscr{S}_\Omega$ can be described in a geometrical intuitive way by control points, whose topological structure is a natural generalization of the tensor product arrangement. Unlike proper control points some of the control points assigned to patches sharing extraordinary vertices cannot be chosen arbitrarily but have to fulfill certain constraints related to the conditions imposed on D-patches. Therefore, they will be referred to as *quasi control points*. In the fourth section, the refinement property of the spline space $\mathscr{S}_\Omega$ is established in terms of a linear map acting on control points. This map is uniform in the sense that it uses the same mask of weights equally on the regular and the extraordinary parts of the surface. In the fifth section, a linear subspace $\mathscr{S}^A_\Omega \subset \mathscr{S}_\Omega$ is specified by replacing the coplanarity condition for quasi control points by fixed linear dependencies. Thus, by providing linearity, one of the major prerequisites for various applications is fulfilled. In the sixth section methods for projecting arbitrarily chosen control points to the space of quasi control points are given. On one hand, this is of particular importance for design purposes where control points are assumed to be manipulatable without restrictions. But on the other hand, this allows the construction of a family of real-valued B-spline functions spanning a space of smooth spline surfaces and providing most of the favorable properties of ordinary B-splines.

1. REGULARITY OF DEGENERATE PATCHES

Consider $n \in \mathbb{N} \backslash \{1, 2\}$ analytic surface segments over the unit square $\omega := [0, 1]^2$,

$$\mathbf{x}^j \colon \omega \ni \mathbf{u} := (u, v) \mapsto \sum_{p,\, q=0}^{\infty} \mathbf{A}^j_{pq} u^p v^q \in \mathbb{R}^3, \qquad j = 1, ..., n, \tag{1.1}$$

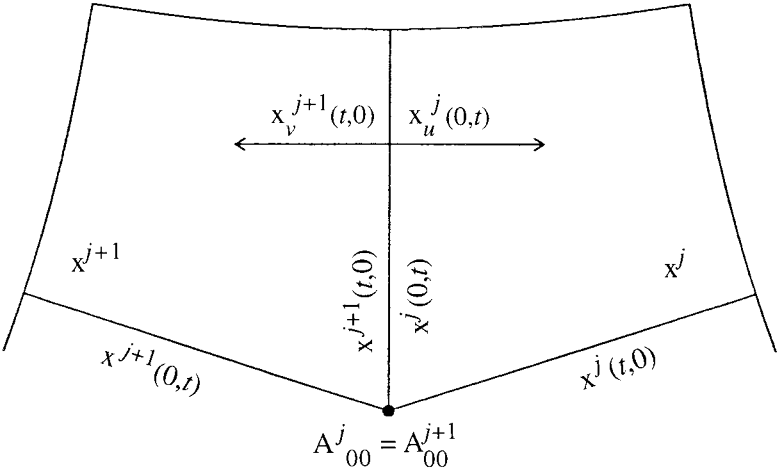


FIG. 1.1. Parametrically smooth contact of two patches.

joining parametrically smoothly according to

$$\mathbf{x}^j(0, t) = \mathbf{x}^{j+1}(t, 0), \qquad \mathbf{x}^j_u(0, t) = -\mathbf{x}^{j+1}_v(t, 0), \qquad t \in [0, t]; \tag{1.2}$$

see Fig. 1.1. Here and subsequently, the index j runs from 1 to n and has to be understood modulo n. By uniqueness of Taylor series, the smoothness conditions imply

$$\mathbf{A}^j_{0r} = \mathbf{A}^{j+1}_{r0}, \qquad \mathbf{A}^j_{1r} = -\mathbf{A}^{j+1}_{r1}, \qquad r \in \mathbb{N}_0. \tag{1.3}$$

For $r \geqslant 2$, these equations are decoupled and no problems arise. However, for $r \in \{0, 1\}$ we obtain a cyclic system of equations,

$$\begin{aligned} \mathbf{A}^j_{00} &= \mathbf{A}^{j+1}_{00}, \qquad & \mathbf{A}^j_{10} &= -\mathbf{A}^{j+1}_{01} \\ \mathbf{A}^j_{01} &= \mathbf{A}^{j+1}_{10}, \qquad & \mathbf{A}^j_{11} &= -\mathbf{A}^{j+1}_{11}. \end{aligned} \tag{1.4}$$

The first equation simply defines the common center $\mathbf{M} := \mathbf{A}^1_{00} = \cdots = \mathbf{A}^n_{00}$, whereas the other three equations lead to

$$\begin{aligned} \mathbf{A}^j_{01} &= -\mathbf{A}^{j+2}_{01} = \mathbf{A}^{j+4}_{01} \\ \mathbf{A}^j_{10} &= -\mathbf{A}^{j+2}_{10} = \mathbf{A}^{j+4}_{10} \\ \mathbf{A}^j_{11} &= -\mathbf{A}^{j+1}_{11} = \mathbf{A}^{j+2}_{11}. \end{aligned} \tag{1.5}$$

For all $n \in \mathbb{N}$, this system has the trivial solution

$$\mathbf{A}^j_{10} = \mathbf{A}^j_{01} = \mathbf{A}^j_{11} = 0, \qquad j = 1, ..., n, \tag{1.6}$$

corresponding to patches $\mathbf{x}^j$, which are degenerate in the sense that they are singularly parametrized at $\mathbf{M}$ and consequently not necessarily smooth. For n even, there exist additional nontrivial solutions, which are 2-periodic for $n/2$ odd and 4-periodic for $n/2$ even. However, taking into account that the geometry of the configuration requires n-periodic solutions, it turns out that only the case $n = 4$ admits reasonable nontrivial solutions. Therefore, it is commonly called the *regular case*. This observation was made frequently before [e.g., 8, 10], and usually one tries to solve this problem by introducing so-called *geometrical smoothness conditions*. They are weaker than (1.2) but still guarantee smooth joints in the sense of differential geometry. The idea of accepting the trivial solution (1.6) can be found in [17, 15, 16], however, without being verified rigorously. To do so we temporarily confine ourselves to the examination of a single singularly parametrized surface segment and start with the following definitions.

DEFINITION 1.1. An analytic surface segment $\mathbf{x}$ of type

$$\mathbf{x} \colon \omega \ni \mathbf{u} := (u, v) \mapsto \sum_{p,\, q = 0}^{\infty} \mathbf{A}_{pq} u^p v^q \in \mathbb{R}^3 \tag{1.7}$$

is *degenerate* if

$$\mathbf{A}_{10} = \mathbf{A}_{01} = \mathbf{A}_{11} = \mathbf{0}. \tag{1.8}$$

A degenerate surface segment $\mathbf{x}$ is called a *D-patch* if there exist constants $\alpha, \delta \in \mathbb{R}$ and $\beta, \gamma \in \mathbb{R}^+$ such that

$$\begin{pmatrix} \mathbf{A}_{21} \\ \mathbf{A}_{12} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \mathbf{A}_{20} \\ \mathbf{A}_{02} \end{pmatrix}. \tag{1.9}$$

A D-patch is said to be *generic* if $\mathbf{A}_{20}$ and $\mathbf{A}_{02}$ are linearly independent.

THEOREM 1.2. *A generic D-Patch* $\mathbf{x}$ *is regular at* $\mathbf{A}_{00}$; *that is, there exists a regular smooth parametrization representing* $\mathbf{x}$, *locally. The tangent plane passing through* $\mathbf{A}_{00}$ *is spanned by* $\mathbf{A}_{20}$ *and* $\mathbf{A}_{02}$.

Proof. Since $\mathbf{A}_{20}$ and $\mathbf{A}_{02}$ are assumed to be linearly independent, one can choose a coordinate system such that $\mathbf{A}_{00} = \mathbf{O}$ is the origin and $(\mathbf{A}_{20}, \mathbf{A}_{02}) = (\mathbf{e}_1, \mathbf{e}_2)$ are the first two unit vectors. Using the fact that the coefficients $\alpha, \beta, \gamma, \delta$ do not depend on the particular choice of coordinates we obtain $\mathbf{A}_{20} = (1, 0, 0)$, $\mathbf{A}_{21} = (\alpha, \beta, 0)$, $\mathbf{A}_{02} = (0, 1, 0)$, $\mathbf{A}_{12} = (\gamma, \delta, 0)$, and

$$\mathbf{x}(\mathbf{u}) =: \begin{pmatrix} x(\mathbf{u}) \\ y(\mathbf{u}) \\ z(\mathbf{u}) \end{pmatrix} = \begin{pmatrix} u^2 + \gamma uv^2 + O(u^2 v + u^3 + v^3 + (u+v)^4) \\ v^2 + \beta u^2 v + O(uv^2 + u^3 + v^3 + (u+v)^4) \\ O(u^3 + v^3 + (u+v)^4) \end{pmatrix}. \tag{1.10}$$

The xy-plane is expected to be the tangent plane at the origin, so we try to represent $\mathbf{x}$ as the graph of some function $h(x, y)$ near the origin, i.e.,

$$(x, y) \mapsto (x, y, h(x, y)) \in \mathbf{x}. \tag{1.11}$$

This is possible if the projection of $\mathbf{x}$ to the xy-plane is locally injective or, equivalently, if the function $\mathbf{x}\colon U_\varepsilon \ni \mathbf{u} \mapsto (x(\mathbf{u}), y(\mathbf{u}))$ is invertible for $U_\varepsilon := \{\mathbf{u} \in \omega\colon \|\mathbf{u}\| < \varepsilon\}$ and ε small enough. Denote the Jacobian of $\mathbf{x}$ by J and the symmetrized Jacobian by

$$\tilde{J} := (J + J^{\mathrm{T}})/2; \tag{1.12}$$

then a short computation yields

$$\begin{aligned} \operatorname{trace} \tilde{J}(\mathbf{u}) &= 2(u+v) + O((u+v)^2) \\ \det \tilde{J}(\mathbf{u}) &= 2(\beta u^3 + \gamma v^3 + 2uv) + O(u^2 v + uv^2 + (u+v)^4). \end{aligned} \tag{1.13}$$

Both expressions are positive for $\mathbf{u} \in U_\varepsilon \backslash (0, 0)$ and ε sufficiently small. For the trace, this is obvious, and for the determinant it can be shown as follows: The inequalities

$$\begin{aligned} \beta u^3 + \gamma v^3 + 2uv &\geqslant 2uv \\ \beta u^3 + \gamma v^3 + 2uv &\geqslant \kappa (u+v)^3 \end{aligned} \tag{1.14}$$

with $\kappa := \min\{\beta, \gamma, \frac{1}{3}\}$ hold for all $\beta, \gamma > 0$ and $\mathbf{u} \in \omega$. Thus,

$$\begin{aligned} \det \tilde{J}(\mathbf{u}) &\geqslant 2uv + O(u^2 v + uv^2) + \kappa (u+v)^3 + O((u+v)^4) \\ &\geqslant (2uv + \kappa (u+v)^3)(1 + O(u+v)). \end{aligned} \tag{1.15}$$

With the trace and determinant being positive, the matrix $\tilde{J}(\mathbf{u})$ is positive definite; i.e.,

$$\mathbf{r} J(\mathbf{u})\, \mathbf{r}^{\mathrm{T}} = \mathbf{r} \tilde{J}(\mathbf{u})\, \mathbf{r}^{\mathrm{T}} > 0 \tag{1.16}$$

for every row-vector $\mathbf{r} \in \mathbb{R}^2 \backslash (0, 0)$ and $\mathbf{u} \in U_\varepsilon \backslash (0, 0)$. Now, in order to show injectivity of $\mathbf{x}$, consider two points $\mathbf{u}_1, \mathbf{u}_2 \in U_\varepsilon$ with $\mathbf{x}(\mathbf{u}_1) = \mathbf{x}(\mathbf{u}_2)$. Define $\mathbf{r} := \mathbf{u}_2 - \mathbf{u}_1$ and

$$s(t) = \mathbf{r}\mathbf{x}(\mathbf{u}_1 + t\mathbf{r}), \qquad t \in [0, 1]; \tag{1.17}$$

then $s(0) = s(1)$. Consequently, by the mean value theorem, there is a $\tau \in (0, 1)$ with

$$s'(\tau) = \mathbf{r} J(\mathbf{u}_1 + \tau \mathbf{r})\, \mathbf{r}^{\mathrm{T}} = 0. \tag{1.18}$$

Either $\mathbf{u}_1 = \mathbf{u}_2 = (0, 0)$ or the argument of the Jacobian is an element of the convex set $U_\varepsilon \backslash (0, 0)$ and $\mathbf{r} = (0, 0)$ by (1.16). Hence, $\mathbf{x}$ is injective for ε sufficiently small and the patch $\mathbf{x}$ can be parametrized near the origin according to (1.11) with

$$h(x, y) := z(\mathbf{x}^{-1}(x, y)), \qquad (x, y) \in \mathbf{x}(U_\varepsilon) =: V_\varepsilon. \tag{1.19}$$

$\mathbf{x}^{-1}$ is continuous on V_ε and, moreover, it is also continuously differentiable on $V_\varepsilon \backslash (0, 0)$. This follows from the inverse function theorem and the inequality

$$\det J(\mathbf{u}) \geqslant \det \tilde{J}(\mathbf{u}) > 0, \qquad \mathbf{u} \in U_\varepsilon \backslash (0, 0), \tag{1.20}$$

where the first estimate is valid for arbitrary 2×2-matrices satisfying (1.12). For the gradient of h we obtain, using the chain rule and (1.15),

$$\begin{aligned} \lim_{(x, y) \to (0, 0)} \|\nabla h(x, y)\| &= \lim_{\mathbf{u} \to (0, 0)} \|\nabla z(\mathbf{u})\, J(\mathbf{u})^{-1}\| \\ &= \lim_{\mathbf{u} \to (0, 0)} \frac{O(u^2 v + u v^2 + (u + v)^4)}{\det J(\mathbf{u})} \\ &\leqslant \lim_{\mathbf{u} \to (0, 0)} \frac{O(u^2 v + u v^2 + (u + v)^4)}{2uv + \kappa (u + v)^3} = 0. \end{aligned} \tag{1.21}$$

As an immediate consequence of the mean value theorem, this implies that h is continuously differentiable on the entire domain V_ε and $\nabla h(0, 0) = (0, 0)$. So, (1.11) is a regular smooth parametrization of $\mathbf{x}$ near the origin and the tangent plane at the origin is spanned by $\mathbf{A}_{20}$ and $\mathbf{A}_{02}$ as stated. ∎

Although not always stated explicitly, D-patches are assumed to be generic throughout this paper. The exceptional case of nongeneric D-patches is of minor importance and not considered here.

The conditions imposed on D-patches are necessary for regularity in the following sense: Assume that $(\mathbf{A}_{20}, \mathbf{A}_{02})$, $(\mathbf{A}_{20}, \mathbf{A}_{21})$, and $(\mathbf{A}_{02}, \mathbf{A}_{12})$ are linearly independent, respectively, and define the normal vector $\mathbf{n}(\mathbf{u})$ of $\mathbf{x}$ at a regularly parametrized point $\mathbf{x}(\mathbf{u})$ by

$$\mathbf{n}(\mathbf{u}) := \frac{\mathbf{x}_u(\mathbf{u}) \times \mathbf{x}_v(\mathbf{u})}{\|\mathbf{x}_u(\mathbf{u}) \times \mathbf{x}_v(\mathbf{u})\|}. \tag{1.22}$$

Then we obtain for different paths of parameters approaching the origin:

$$\begin{aligned} \lim_{t \downarrow 0} \mathbf{n}(t, t) &= \frac{\mathbf{A}_{20} \times \mathbf{A}_{02}}{\|\mathbf{A}_{20} \times \mathbf{A}_{02}\|} \\ \lim_{t \downarrow 0} \mathbf{n}(t, 0) &= \frac{\mathbf{A}_{20} \times \mathbf{A}_{21}}{\|\mathbf{A}_{20} \times \mathbf{A}_{21}\|} \\ \lim_{t \downarrow 0} \mathbf{n}(0, t) &= \frac{\mathbf{A}_{12} \times \mathbf{A}_{02}}{\|\mathbf{A}_{12} \times \mathbf{A}_{02}\|}. \end{aligned} \tag{1.23}$$

Regularity implies equality of all three expressions, and so, obviously, $\mathbf{A}_{20}$, $\mathbf{A}_{02}$, $\mathbf{A}_{21}$, and $\mathbf{A}_{12}$ must be coplanar. Using the representation (1.9) we obtain

$$\lim_{t \downarrow 0} \mathbf{n}(t, 0) = \operatorname{sign}(\beta) \frac{\mathbf{A}_{20} \times \mathbf{A}_{02}}{\|\mathbf{A}_{20} \times \mathbf{A}_{02}\|}, \qquad \lim_{t \downarrow 0} \mathbf{n}(0, t) = \operatorname{sign}(\gamma) \frac{\mathbf{A}_{20} \times \mathbf{A}_{02}}{\|\mathbf{A}_{20} \times \mathbf{A}_{02}\|} \tag{1.24}$$

and, consequently, β and γ must be positive.

Considering the main curvatures κ_1, κ_2 of D-patches it turns out that, in general, they diverge near the singular point. A suitable measure for the rate of divergence, which is independent of the particular parametrization, is the set of exponents $p \in \mathbb{R}^+$ for which $|\kappa_{1,2}|^p$ is integrable over $\mathbf{x}$ near the singular point.

Definition 1.3. The seminorm $\|\cdot\|_{p,\varepsilon}$, $p \geqslant 1$, $\varepsilon > 0$, for functions f over $\mathbf{x}$ is given by

$$\|f\|_{p,\varepsilon} := \left(\int_{\mathbf{x}_\varepsilon} |f|^p \, dS \right)^{1/p}, \tag{1.25}$$

where $\mathbf{x}_\varepsilon$ denotes the restriction of $\mathbf{x}$ to the domain U_ε.

Theorem 1.4. *$\|\kappa_i\|_{p,\varepsilon}$, $i \in \{1, 2\}$, is finite for $1 \leqslant p < 4$ and ε sufficiently small.*

Proof. A detailed proof of this theorem is rather technical without revealing deeper insight into the problem. So we confine ourselves to an outline of the crucial estimates. Consider the patch $\mathbf{x}$ in canonical form (1.10) and introduce polar coordinates according to $(u, v) = r(\cos(t), \sin(t))$, $(r, t) \in [0, \varepsilon) \times [0, \pi/2]$. Then we obtain for the first fundamental form

$$g := \begin{pmatrix} \mathbf{x}_u \cdot \mathbf{x}_u & \mathbf{x}_u \cdot \mathbf{x}_v \\ \mathbf{x}_u \cdot \mathbf{x}_v & \mathbf{x}_v \cdot \mathbf{x}_v \end{pmatrix} = O(r^2) \tag{1.26}$$

and, using the inequalities (1.14),

$$\begin{aligned} \det g &= \|\mathbf{x}_u \times \mathbf{x}_v\|^2 = O(r^4) \\ (\det g)^{-1} &= O(1/r^4). \end{aligned} \tag{1.27}$$

By (1.10), the normal vector $\mathbf{n}$ converges to the third unit vector $\mathbf{e}_3$ according to $\mathbf{n} - \mathbf{e}_3 = O(r)$. This implies for the second fundamental form

$$h := \begin{pmatrix} \mathbf{n} \cdot \mathbf{x}_{uu} & \mathbf{n} \cdot \mathbf{x}_{uv} \\ \mathbf{n} \cdot \mathbf{x}_{uv} & \mathbf{n} \cdot \mathbf{x}_{vv} \end{pmatrix} = O(r). \tag{1.28}$$

Now, we find for the mean curvature H and the Gaussian curvature K,

$$\begin{aligned} 2H &= \kappa_1 + \kappa_2 = \operatorname{trace}(hg^{-1}) = O(1/r) \\ K &= \kappa_1 \kappa_2 = \det h / \det g = O(1/r^2), \end{aligned} \tag{1.29}$$

and consequently, the main curvatures are of order $\kappa_i = O(1/r)$, $i \in \{1, 2\}$. Finally, we obtain

$$\|\kappa_i\|_{p,\varepsilon}^p = \int_0^{\pi/2} \int_0^\varepsilon |\kappa_i|^p \, \|\mathbf{x}_u \times \mathbf{x}_v\| \, r \, dr \, dt = \int_0^\varepsilon O(r^{3-p}) \, dr \tag{1.30}$$

and finiteness of the integral for $1 \leqslant p < 4$. As can be shown by examples, the given range of exponents is sharp in the sense that the integral is not necessarily finite for $p = 4$. ∎

2. DEGENERATE BÉZIER PATCHES

The results derived in the preceding section suggest that D-patches are suitable for generating smooth surfaces using various types of basis functions, including polynomial, rational, trigonometric, or exponential splines.

However, the further development of the theory will be restricted to a case of particular interest, namely modeling surfaces by polynomial patches.

As a common practice in computer-aided design we replace the monomial representation (1.7) of a polynomial patch by the more convenient *Bernstein–Bézier* form (2.2). To this end denote the Bernstein polynomials of degree $d \geqslant 2$ by

$$\begin{aligned} b_d^p(u) &:= \binom{d}{p} u^p (1-u)^{d-p}, \qquad p = 0, ..., d, \\ b_d(u) &:= [b_d^0(u), ..., b_d^d(u)]. \end{aligned} \tag{2.1}$$

The *Bézier patch* $\mathbf{x_B}$ corresponding to a $(d+1)\times(d+1)$-matrix $\mathbf{B}$ of *Bézier points* $\mathbf{B}_{pq} \in \mathbb{R}^3$ is defined by

$$\mathbf{x_B}(u, v) = b_d(u)\, \mathbf{B} b_d(v)^{\mathrm{T}}, \qquad (u, v) \in [0, 1]^2. \tag{2.2}$$

The following lemma provides necessary and sufficient criteria for Bézier patches matching Definition 1.1. Here and subsequently, it is assumed that the singular point is located at $(u, v) = (0, 0)$. Due to the inherent symmetries of the Bernstein–Bézier representation this is no loss of generality.

LEMMA 2.1. *A Bézier patch $\mathbf{x_B}$ is degenerate if and only if the Bézier points $\mathbf{B}_{00} = \mathbf{B}_{10} = \mathbf{B}_{01} = \mathbf{B}_{11}$ coalesce. $\mathbf{x_B}$ is a D-patch if and only if, in addition, there exist constants $\alpha, \delta \in \mathbb{R}$ and $\beta, \gamma \in \mathbb{R}^+$ such that*

$$\begin{pmatrix} \mathbf{B}_{21} - \mathbf{B}_{00} \\ \mathbf{B}_{12} - \mathbf{B}_{00} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \mathbf{B}_{20} - \mathbf{B}_{00} \\ \mathbf{B}_{02} - \mathbf{B}_{00} \end{pmatrix} \tag{2.3}$$

Proof. Expanding (2.2) yields

$$\begin{aligned} \mathbf{x_B}(u, v) = \mathbf{B}_{00} &+ d((\mathbf{B}_{10} - \mathbf{B}_{00})\, u + (\mathbf{B}_{01} - \mathbf{B}_{00})\, v) \\ &+ d^2(\mathbf{B}_{11} + \mathbf{B}_{00} - \mathbf{B}_{10} - \mathbf{B}_{01})\, uv + \text{h.o.t.}, \end{aligned} \tag{2.4}$$

and comparison with (1.8) shows that $\mathbf{x_B}$ is degenerate if and only if $\mathbf{B}_{00} = \mathbf{B}_{10} = \mathbf{B}_{01} = \mathbf{B}_{11}$. Using these identities, we obtain from (2.2)

$$\begin{aligned} \mathbf{x_B} = \mathbf{B}_{00} &+ d(d-1)((\mathbf{B}_{20} - \mathbf{B}_{00})\, u^2 + (\mathbf{B}_{02} - \mathbf{B}_{00})\, v^2)/2 \\ &+ d^2(d-1)(\mathbf{B}_{21} - \mathbf{B}_{20})\, u^2 v + (\mathbf{B}_{12} + \mathbf{B}_{02})\, uv^2)/2 + \text{h.o.t.}, \end{aligned} \tag{2.5}$$

and $\mathbf{x_B}$ is a D-patch if and only if there exist constants $\alpha, \delta \in \mathbb{R}$ and $\beta, \gamma \in \mathbb{R}^+$ with

$$d^2(d-1) \begin{pmatrix} \mathbf{B}_{21} - \mathbf{B}_{20} \\ \mathbf{B}_{12} - \mathbf{B}_{02} \end{pmatrix} = d(d-1) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \mathbf{B}_{20} - \mathbf{B}_{00} \\ \mathbf{B}_{02} - \mathbf{B}_{00} \end{pmatrix}. \tag{2.6}$$

The latter equation is equivalent to

$$\begin{pmatrix} \mathbf{B}_{21}-\mathbf{B}_{00} \\ \mathbf{B}_{12}-\mathbf{B}_{00} \end{pmatrix} = \begin{pmatrix} \mathbf{B}_{21}-\mathbf{B}_{20} \\ \mathbf{B}_{12}-\mathbf{B}_{02} \end{pmatrix} + \begin{pmatrix} \mathbf{B}_{20}-\mathbf{B}_{00} \\ \mathbf{B}_{02}-\mathbf{B}_{00} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha/d+1 & \beta/d \\ \gamma/d & \delta/d+1 \end{pmatrix} \begin{pmatrix} \mathbf{B}_{20}-\mathbf{B}_{00} \\ \mathbf{B}_{02}-\mathbf{B}_{00} \end{pmatrix} \tag{2.7}$$

and renaming the matrix entries by $\alpha, \beta, \gamma, \delta$, respectively, completes the proof. ▮

COROLLARY 2.2. *The tangent plane of a D-patch in Bernstein–Bézier form* (2.2) *at the singular point is determined by* $\mathbf{B}_{00}$, $\mathbf{B}_{20}$, *and* $\mathbf{B}_{02}$.

Degenerate bilinear patches were excluded a priori since, evidently, they are shrunk to single points rather than being proper surface segments. But there is a further unwelcome phenomenon raising the bi-degree d which is necessary for generating reasonable D-patches. If $d=2$ the boundary curves $\mathbf{x_B}(t, 0)$ and $\mathbf{x_B}(0, t)$ emanating from the singular point degenerate to straight lines, thus restricting the shape of patches inadmissibly. In general, choosing $d \geqslant 3$ is sufficient to avoid this effect, except for one case. It is conceivable that a patch has not only one but several singular points. If $d=3$ and if besides $\mathbf{x_B}(0, 0)$ also one of the adjacent corners $\mathbf{x_B}(1, 0)$ or $\mathbf{x_B}(0, 1)$ is singular then the boundary curve connecting the singular points is a straight line, again.

3. A SPLINE SPACE INCORPORATING D-PATCHES

In order to obtain smooth surfaces a spline space must be provided with a set of smoothness conditions. One part of these smoothness conditions requires the segments to be parametrized by smooth functions. The other and less trivial part establishes rules how to join adjacent patches. The spline space $\mathcal{S}_\Omega$ to be considered here is defined over some semiregular domain Ω and the patches are bicubic polynomials joining parametrically smoothly, according to (1.2). In view of the observations made in the preceding section, this choice is the simplest one, avoiding unwelcome geometrical restrictions, and is merely made for the sake of clarity. Generalizations to irregular domains and patches of arbitrary degree are straightforward. By the way, any irregular domain can be made semiregular by splitting each subdomain into four quadratic pieces. So, semiregularity is not an essential restriction.

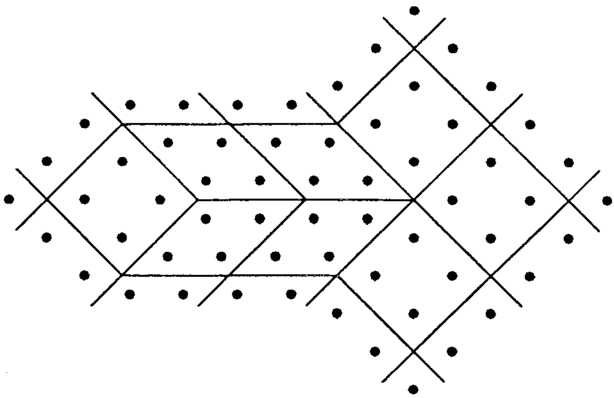


FIG. 3.1. Domain Ω—structure of patches and control points.

An important property of $\mathscr{S}_\Omega$ is that it can be represented by a set of control points. They are arranged according to the structure of patches (see Fig. 3.1), and give an intuitive geometrical description of the corresponding spline surface. Thus, one of the major benefits of modeling with B-splines is preserved. Dealing with bicubic patches joining with smoothness of first order it seems reasonable to choose control points compatibly to bicubic tensor product B-splines with equally spaced double knots. First, this means that four control points are assigned to the interior of each patch. Second, passing from control points to Bézier form is characterized by the fact that the control points themselves are Bézier points from which the remaining ones assigned to the edges are simply computed by averaging of direct neighbors. This arrangement of control points turns out to be equally suitable for representing $\mathscr{S}_\Omega$ despite of its higher combinatorial complexity. To state this fact more precisely we start with considering a regular patch $\mathbf{x}^i$; i.e., $\mathbf{x}^i$ incorporates no extraordinary vertices. The corresponding 4×4-matrix $\mathbf{C}^i$ of control points is assembled by the control points assigned to $\mathbf{x}^i$ and its eight neighbors as indicated by Fig. 3.2.

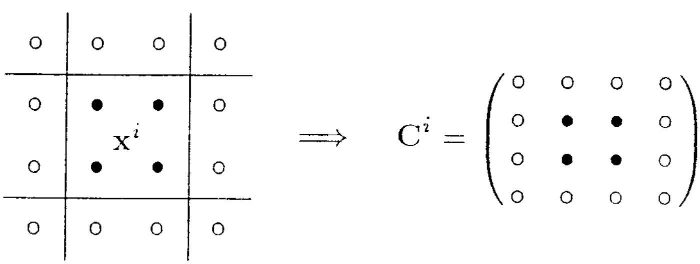


FIG. 3.2. Matrix $\mathbf{C}^i$—regular case.

Denote by $c_3(u) := [c_3^0(u), ..., c_3^3(u)]$ the vector of the four cubic B-splines supported on $[0, 1]$ with double knots at the integers. Then the tensor product spline surface corresponding to $\mathbf{C}^i$ is

$$\mathbf{x}^i(u, v) = c_3(u)\, \mathbf{C}^i c_3(v)^{\mathrm{T}};$$

see [11] for a comprehensive introduction to B-spline theory. In order to represent $\mathbf{x}^i$ in Bernstein–Bézier form,

$$\mathbf{x}^i(u, v) = b_3(u)\, \mathbf{B}^i b_3(v)^{\mathrm{T}},$$

according to (2.2), $\mathbf{C}^i$ can be transformed to the matrix $\mathbf{B}^i$ of Bézier points by means of de Boor's knot insertion algorithm [11]. The rule turns out to be particularly simple,

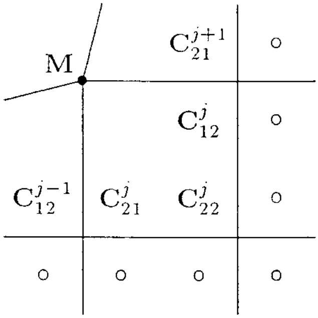
$$\mathbf{B}^i := A\mathbf{C}^i A^{\mathrm{T}}, \qquad A := \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}. \tag{3.1}$$

As mentioned above, this means that the four inner control points remain unchanged, whereas the remaining ones are obtained by averaging. Near an extraordinary vertex of order n the situation is only slightly different. Again, four control points $\{\mathbf{C}_{11}^j, \mathbf{C}_{12}^j, \mathbf{C}_{21}^j, \mathbf{C}_{22}^j\}$ are assigned to each of the n patches $\mathbf{x}^j$, $j = 1, ..., n$, and as in the regular case they are assumed to coincide with the inner Bézier points of the corresponding patch. According to the results of the first section, the patches $\mathbf{x}^j$ must be bicubic D-patches, implying that the control points cannot be chosen completely arbitrarily. Unlike ordinary control points they have to satisfy certain constraints stemming from Lemma 2.1. In order to emphasize this difference the points $\mathbf{C}_{11}^j, \mathbf{C}_{21}^j, \mathbf{C}_{12}^j$ will also be referred to as *quasi control points* in contrast to *proper control points* which are free of restrictions. First, Lemma 2.1 enforces that the n innermost quasi control points coalesce, i.e.,

$$\mathbf{C}_{11}^1 = \cdots = \mathbf{C}_{11}^n =: \mathbf{M}. \tag{3.2}$$

Exploiting this fact, the smooth joint of two adjacent patches $\mathbf{x}^j, \mathbf{x}^{j+1}$ according to (1.2) can be achieved using quite the same averaging process as in the regular case. Defining the 4×4-matrices $\mathbf{C}^j$ as indicated by Fig. 3.3 the matrices $\mathbf{B}^j$ of Bézier points are given again by

$$\mathbf{B}^j := A\mathbf{C}^j A^{\mathrm{T}} = \begin{pmatrix} \mathbf{M} & \mathbf{M} & (\mathbf{C}_{12}^j + \mathbf{C}_{21}^{j+1})/2 & \circ \\ \mathbf{M} & \mathbf{M} & \mathbf{C}_{12}^j & \circ \\ (\mathbf{C}_{21}^j + \mathbf{C}_{12}^{j-1})/2 & \mathbf{C}_{21}^j & \mathbf{C}_{22}^j & \circ \\ \circ & \circ & \circ & \circ \end{pmatrix}. \tag{3.3}$$



$$\Longrightarrow \quad \mathbf{C}^j = \begin{pmatrix} \mathbf{M} & \mathbf{M} & \mathbf{C}_{21}^{j+1} & \circ \\ \mathbf{M} & \mathbf{M} & \mathbf{C}_{12}^{j} & \circ \\ \mathbf{C}_{12}^{j-1} & \mathbf{C}_{21}^{j} & \mathbf{C}_{22}^{j} & \circ \\ \circ & \circ & \circ & \circ \end{pmatrix}$$

FIG. 3.3. Matrix $\mathbf{C}^j$—extraordinary case.

Note that the definition of $\mathbf{C}^j$ differs from the regular case only by the somewhat artificial assignment of the fourfold entry $\mathbf{M}$. Comparison of (3.3) with Lemma 2.1 yields

COROLLARY 3.1. *The quasi control points* $\mathbf{C}_{11}^j, \mathbf{C}_{12}^j, \mathbf{C}_{21}^j$ *must satisfy the following constraints*:

(i) *The points* $\mathbf{C}_{11}^1 = \cdots = \mathbf{C}_{11}^n =: \mathbf{M}$ *coalesce.*

(ii) *There exist constants* $\alpha^j, \delta^j \in \mathbb{R}$ *and* $\beta^j, \gamma^j \in \mathbb{R}^+$ *such that for all* $j = 1, ..., n$

$$\begin{pmatrix} \mathbf{C}_{21}^j - \mathbf{M} \\ \mathbf{C}_{12}^j - \mathbf{M} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \alpha^j & \beta^j \\ \gamma^j & \delta^j \end{pmatrix} \begin{pmatrix} \mathbf{C}_{21}^j + \mathbf{C}_{12}^{j-1} - 2\mathbf{M} \\ \mathbf{C}_{21}^{j+1} + \mathbf{C}_{12}^j - 2\mathbf{M} \end{pmatrix}. \tag{3.4}$$

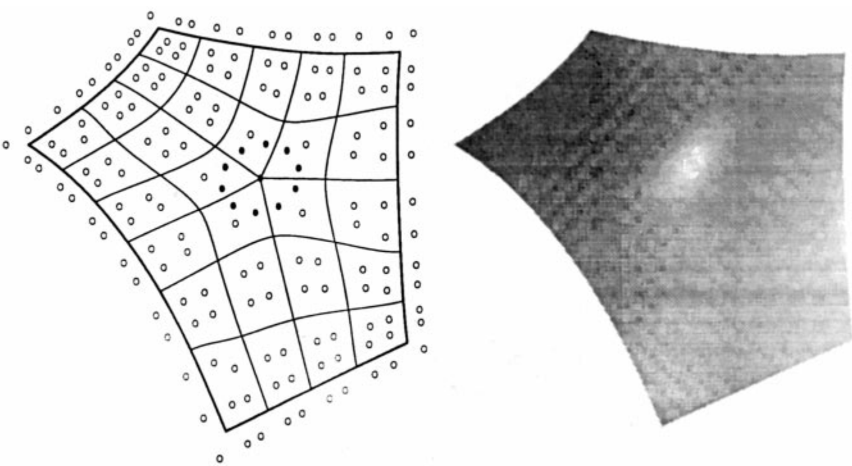


FIG. 3.4. Spline surface near an extraordinary vertex of order $n = 5$.

A typical example illustrating the theory developed in this section is given in Fig. 3.4. On the left-hand side, one sees control points ($\circ$) and quasi control points ($\bullet$), together with the patch boundaries near an extraordinary vertex of order $n=5$. On the right-hand side, the corresponding shaded spline surface is shown.

4. THE REFINEMENT PROPERTY

The major benefit of the approach presented in the previous sections is that the spline space $\mathscr{S}_\Omega$ is refineable in the following sense: A refined domain $\tilde{\Omega}$ is obtained by splitting every subdomain (ω, i) of Ω into four smaller squares and then rescaling the new subdomains to original size. Thus, the original domain $\Omega = \omega \times I$ is transformed to $\tilde{\Omega} = \omega \times (I \times \{1, 2, 3, 4\})$. It is provided with the new connectivity relation $\tilde{\mathscr{C}}$ which is simply obtained from $\mathscr{C}$ by two steps. First, each pair of neighbors is converted into two pairs according to the split of the corresponding subdomains. Second, the relations between any four new subdomains stemming from the split of the same original subdomain have to be added. The spline space $\mathscr{S}_\Omega$ and its refinement $\mathscr{S}_{\tilde{\Omega}}$ are similar in the sense that number and order of the extraordinary vertices coincide; that is, all new inserted vertices are regular. The refinement process is illustrated in Fig. 4.1.

The generation of $\mathscr{S}_{\tilde{\Omega}}$ from $\mathscr{S}_\Omega$ is exclusively based on topological facts and actually rather trivial. However, and this is the crucial point, there exists an induced analytical transformation acting on spline functions which is characterized by the invariance of graphs.

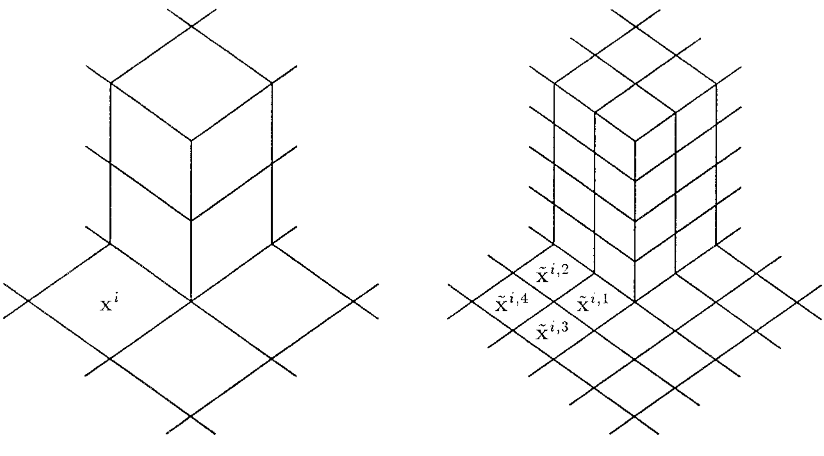


FIG. 4.1. Refining a domain $\Omega \mapsto \tilde{\Omega}$.

THEOREM 4.1. *There exists a canonical embedding* $F: \mathcal{S}_\Omega \hookrightarrow \mathcal{S}_{\tilde{\Omega}}$ *such that for all spline surfaces* $\mathbf{x} \in \mathcal{S}_\Omega$ *and* $\tilde{\mathbf{x}} := F(\mathbf{x}) \in \mathcal{S}_{\tilde{\Omega}}$ *the graphs* $\mathbf{x}(\Omega)$ *and* $\tilde{\mathbf{x}}(\tilde{\Omega})$ *coincide when regarded as point sets in* $\mathbb{R}^3$.

Proof. The proof is constructive. Consider a segment

$$\mathbf{x}^i: (u, v) \mapsto b_3(u)\, \mathbf{B}^i b_3(v)^{\mathrm{T}} \tag{4.1}$$

of the spline surface $\mathbf{x} \in \mathcal{S}_\Omega$. Then the four corresponding segments,

$$\begin{aligned}
\tilde{\mathbf{x}}^{i,1}&: (u, v) \mapsto b_3(u)\, \tilde{\mathbf{B}}^{i,1} b_3(v)^{\mathrm{T}} := \mathbf{x}^i(u/2, v/2)\\
\tilde{\mathbf{x}}^{i,2}&: (u, v) \mapsto b_3(u)\, \tilde{\mathbf{B}}^{i,2} b_3(v)^{\mathrm{T}} := \mathbf{x}^i((u+1)/2, v/2)\\
\tilde{\mathbf{x}}^{i,3}&: (u, v) \mapsto b_3(u)\, \tilde{\mathbf{B}}^{i,3} b_3(v)^{\mathrm{T}} := \mathbf{x}^i(u/2, (v+1)/2)\\
\tilde{\mathbf{x}}^{i,4}&: (u, v) \mapsto b_3(u)\, \tilde{\mathbf{B}}^{i,4} b_3(v)^{\mathrm{T}} := \mathbf{x}^i((u+1)/2, (v+1)/2),
\end{aligned} \tag{4.2}$$

of $\tilde{\mathbf{x}} := F(\mathbf{x})$ are generated by the well-known procedure of subdividing Bézier patches [11]. By symmetry, it is sufficient to specify only one formula for computing $\tilde{\mathbf{B}}^{i,k}$, e.g.,

$$\tilde{\mathbf{B}}^{i,1} := S\mathbf{B}^i S^{\mathrm{T}}, \qquad S := \begin{pmatrix} 1 & 0 & 0 & 0\\ 1/2 & 1/2 & 0 & 0\\ 1/4 & 1/2 & 1/4 & 0\\ 1/8 & 3/8 & 3/8 & 1/8 \end{pmatrix}. \tag{4.3}$$

By construction, $\bigcup_k \tilde{\mathbf{x}}^{i,k}(\omega) = \mathbf{x}^i(\omega)$ and, thus, applying subdivision to all patches implies $\tilde{\mathbf{x}}(\tilde{\Omega}) = \mathbf{x}(\Omega)$ as required. Further, it is evident that the new patches $\tilde{\mathbf{x}}^{i,k}$ join parametrically smoothly. So it remains to show that subdivision applied to D-patches yields D-patches, again. To this end consider n D-patches $\mathbf{x}^j$ sharing the extraordinary point $\mathbf{M}$. Specifying only the relevant entries of the subdivided matrix of Bézier points we find

$$\tilde{\mathbf{B}}^{j,1} = \frac{1}{8}\begin{pmatrix} 8\mathbf{M} & 8\mathbf{M} & 6\mathbf{M} + 2\mathbf{B}^j_{02} & \circ\\ 8\mathbf{M} & 8\mathbf{M} & 6\mathbf{M} + \mathbf{B}^j_{02} + \mathbf{B}^j_{12} & \circ\\ 6\mathbf{M} + 2\mathbf{B}^j_{20} & 6\mathbf{M} + \mathbf{B}^j_{20} + \mathbf{B}^j_{21} & \circ & \circ\\ \circ & \circ & \circ & \circ \end{pmatrix}. \tag{4.4}$$

The four entries in the upper left corner coincide and a short computation shows that

$$\begin{pmatrix} \mathbf{B}^j_{21} - \mathbf{M}\\ \mathbf{B}^j_{12} - \mathbf{M} \end{pmatrix} = \begin{pmatrix} \alpha^j & \beta^j\\ \gamma^j & \delta^j \end{pmatrix} \begin{pmatrix} \mathbf{B}^j_{20} - \mathbf{M}\\ \mathbf{B}^j_{02} - \mathbf{M} \end{pmatrix} \tag{4.5}$$

implies

$$\begin{pmatrix} \tilde{\mathbf{B}}^j_{21} - \mathbf{M} \\ \tilde{\mathbf{B}}^j_{12} - \mathbf{M} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \alpha^j + 1 & \beta^j \\ \gamma^j & \delta^j + 1 \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{B}}^j_{20} - \mathbf{M} \\ \tilde{\mathbf{B}}^j_{02} - \mathbf{M} \end{pmatrix}. \tag{4.6}$$

Consequently, the coefficients in question are positive and $\tilde{\mathbf{x}}^{j,1}$ is a D-patch. ∎

As discussed in the preceding section, switching between the representation by Bézier points and control points is particularly simple. In one direction, it is done by averaging and in the other direction by deleting the Bézier points assigned to the edges. Thus, the refinement process described in terms of Bézier points has a counterpart in the space of control points. Starting with the original control points the corresponding Bézier points have to be computed first by (3.1). Then subdivision is carried out in the space of Bézier points according to (4.3), and finally the new control points are obtained by inverting (3.1). Since (3.1) is equally valid for regular and extraordinary patches we obtain a single subdivision formula combining these three steps,

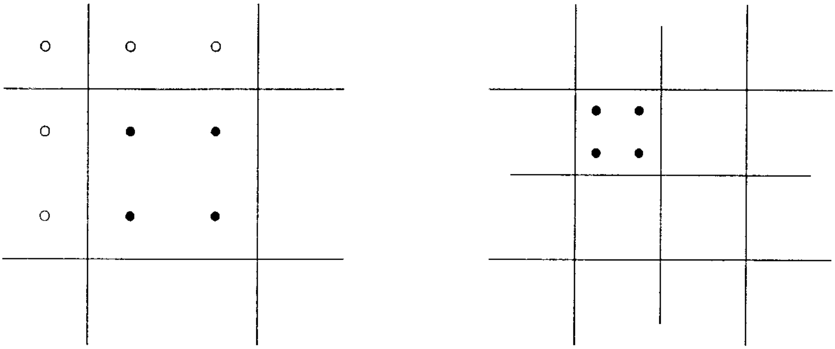
$$\tilde{\mathbf{C}}^{i,1} = T\mathbf{C}^i T^{\mathrm{T}}, \qquad T := A^{-1}SA = \frac{1}{8}\begin{pmatrix} 6 & 2 & 0 & 0 \\ 2 & 6 & 0 & 0 \\ 1 & 5 & 2 & 0 \\ 0 & 2 & 5 & 1 \end{pmatrix}. \tag{4.7}$$

Due to symmetry, the corresponding expressions for $\tilde{\mathbf{C}}^{i,2}$, $\tilde{\mathbf{C}}^{i,3}$, $\tilde{\mathbf{C}}^{i,4}$ need not be specified explicitly. Actually, (4.7) contains a lot of redundancy, since applying it to all patches will cause multiple evaluation of many new control points. Evidently, it is sufficient to restrict the result of (4.7) to the inner 2×2-submatrix $\tilde{\mathbf{C}}^i_{2:3}$ of $\tilde{\mathbf{C}}^i$ which contains only those new control points assigned to the patch $\tilde{\mathbf{x}}^{i,1}$. Moreover, it turns out that $\tilde{\mathbf{C}}^i_{2:3}$ depends only on the upper left 3×3-submatrix $\mathbf{C}^i_{1:3}$ of $\mathbf{C}^i$ and we obtain the reduced formula,

$$\tilde{\mathbf{C}}^{i,1}_{2:3} = T_r \mathbf{C}^i_{1:3} T_r^{\mathrm{T}}, \qquad T_r := \frac{1}{8}\begin{pmatrix} 2 & 6 & 0 \\ 1 & 5 & 2 \end{pmatrix} \qquad \text{(Fig. 4.2)}. \tag{4.8}$$

Finally, some remarks:

- When the subdivision algorithm is iterated it will generate a sequence $\mathbf{C}_m$, $m \in \mathbb{N}$, of control nets converging to the originally defined spline surface $\mathbf{x}$. The rate of convergence is the same as for repeated subdivision of bicubic Bézier patches, namely $O(4^{-m})$ [3, 12].
- The subdivision algorithm presented here is the first one working on meshes of arbitrary topology which converges to an explicitly known limit.



$$\mathbf{C}^{i}_{1:3} \longmapsto \tilde{\mathbf{C}}^{i,1}_{2:3}$$

FIG. 4.2. Application of the reduced subdivision formula (4.8).

• It is a noteworthy fact that the subdivision formula (4.8) is equally valid for regular and degenerate patches. This uniformity should be advantageous for implementations.

5. A LINEAR SUBSPACE

The spline space $\mathscr{S}_\Omega$ as defined in the third section reveals two drawbacks stemming from the conditions imposed on quasi control points. The first is that these conditions are not linear implying that $\mathscr{S}_\Omega$ is a nonlinear space. This could be a serious problem for further conceivable developments such as approximation of surfaces, constructing wavelets or applications in the theory of finite elements. The second concerns the fact that the sheer presence of conditions is a fundamental obstacle to applications in computer aided design systems. Methods for overcoming these difficulties will be presented in this and the next section.

The problem of nonlinearity can be solved by identifying a linear subspace $\mathscr{S}^A_\Omega \subset \mathscr{S}_\Omega$. To this end the conditions listed in Corollary 3.1 must be modified. The first condition is linear and can be kept. The second condition becomes linear when the constants $\alpha^j, \beta^j, \gamma^j, \delta^j$ are not merely assumed to exist but are specified explicitly and fixed. Thus, these constants play the role of shape parameters, for instance just like relative knot spacings in B-spline spaces. When choosing the constants $\alpha^j, \beta^j, \gamma^j, \delta^j$ one must take into consideration that they have to satisfy a certain consistency condition related to the periodic structure of (3.4).

LEMMA 5.1. *Assume that a set of quasi control points satisfies the constraints specified in Corollary* 3.1. *Then*

$$\prod_{j=1}^{n} \Phi^j = E, \qquad \Phi^j := \frac{1}{\beta^j} \begin{pmatrix} \bar{\alpha}^j & 1 \\ \bar{\alpha}^j \bar{\delta}^j - \beta^j \gamma^j & \bar{\delta}^j \end{pmatrix}, \tag{5.1}$$

with $\bar{\alpha}^j := 1 - \alpha^j$, $\bar{\delta}^j := 1 - \delta^j$, *$E$ denoting the identity, and the product sign addressing multiplication from the left.*

Proof. Introducing the quantities

$$\mathbf{p}^j := \begin{pmatrix} \mathbf{C}^j_{21} + \mathbf{C}^{j-1}_{12} - 2\mathbf{M} \\ \mathbf{C}^j_{21} - \mathbf{C}^{j-1}_{12} \end{pmatrix}, \tag{5.2}$$

a short computation shows that (3.4) implies $\mathbf{p}^{j+1} = \Phi^j \mathbf{p}^j$. Thus, by periodicity, $\mathbf{p}^1 = \mathbf{p}^{n+1} = \prod_{j=1}^{n} \Phi^j \mathbf{p}^1$. Since

$$\mathbf{p}^1 = \begin{pmatrix} 1 & 0 \\ -\bar{\alpha}^1 & \beta^1 \end{pmatrix} \begin{pmatrix} \mathbf{C}^1_{21} + \mathbf{C}^0_{12} - 2\mathbf{M} \\ \mathbf{C}^2_{21} + \mathbf{C}^1_{12} - 2\mathbf{M} \end{pmatrix}, \tag{5.3}$$

the two vectors forming $\mathbf{p}^1$ are linearly independent if the patch $\mathbf{x}^1$ is generic and, consequently, the product of the matrices Φ^j has to be the identity. ▮

DEFINITION 5.2. A set of real constants $\lambda := \{\alpha^j, \beta^j, \gamma^j, \delta^j, j = 1, ..., n\}$ is called *feasible* if $\beta^j, \gamma^j > 0$ and if $\prod_{j=1}^{n} \Phi^j = E$. If $\mathcal{S}_\Omega$ is a spline space incorporating m extraordinary vertices of order n_μ, $\mu = 1, ..., m$ the subspace $\mathcal{S}^\Lambda_\Omega$ is characterized by m feasible sets of constants $\Lambda := \{\lambda_1, ..., \lambda_m\}$ via the following constraints on quasi control points:

(i) For all $\mu = 1, ..., m$ the points $\mathbf{C}^1_{\mu,11} = \cdots = \mathbf{C}^{n_\mu}_{\mu,11} =: \mathbf{M}_\mu$ coalesce.

(ii) For all $\mu = 1, ..., m$ the points $\mathbf{C}^j_{\mu,21}, \mathbf{C}^j_{\mu,12}$ form a solution of the periodic system

$$\begin{pmatrix} \mathbf{C}^j_{\mu,21} - \mathbf{M}_\mu \\ \mathbf{C}^j_{\mu,12} - \mathbf{M}_\mu \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \alpha^j_\mu & \beta^j_\mu \\ \gamma^j_\mu & \delta^j_\mu \end{pmatrix} \begin{pmatrix} \mathbf{C}^j_{\mu,21} + \mathbf{C}^{j-1}_{\mu,12} - 2\mathbf{M}_\mu \\ \mathbf{C}^j_{\mu,12} + \mathbf{C}^{j+1}_{\mu,21} - 2\mathbf{M}_\mu \end{pmatrix}, \qquad j = 1, ..., n_\mu. \tag{5.4}$$

THEOREM 5.3. *$\mathcal{S}^\Lambda_\Omega \subset \mathcal{S}_\Omega$ is a linear space of dimension*

$$\dim \mathcal{S}^\Lambda_\Omega = 3\left(4 \# I - 3 \sum_{\mu=1}^{m} (n_\mu - 1)\right), \tag{5.5}$$

where $\# I$ denotes the total number of patches.

Proof. Evidently, $\mathscr{S}_\Omega^\Lambda$ is linear. The expression in brackets is the number of scalar degrees of freedom which must be multiplied by three since the coefficients lie in $\mathbb{R}^3$. $4\#I$ is the total number of control points and, thus, it has to be shown that $3(n_\mu - 1)$ is the number of linearly independent conditions assigned to an extraordinary vertex of order n_μ. Clearly, the first constraint yields $n_\mu - 1$ independent conditions. As shown in the proof of Lemma 5.1 the second constraint is equivalent to $\mathbf{p}^{j+1} = \Phi^j \mathbf{p}^j$, providing $2n_\mu$ equations for the $2n_\mu$ unknowns $\mathbf{p}^j$. However, exactly two of them can be chosen arbitrarily. If, for instance, $\mathbf{p}^1$ is given then the other variables are uniquely determined by the iteration $\mathbf{p}^{j+1} = \Phi^j \mathbf{p}^j$, $j = 1, ..., n-1$. The remaining equation $\mathbf{p}^1 = \Phi^n \mathbf{p}^n$ is fulfilled automatically, since

$$\Phi^n \mathbf{p}^n = \Phi^n \prod_{j=1}^{n-1} \Phi^j \mathbf{p}^1 = \prod_{j=1}^{n} \Phi^j \mathbf{p}^1 = \mathbf{p}^1. \tag{5.6}$$

Thus, the rank of the second constraint is $2n_\mu - 2$ and the total effective number of conditions is $3n_\mu - 3$ as required. ∎

From a topological point of view the structure of patches forming a spline surface reveals certain local symmetries near an extraordinary vertex of order n. Roughly speaking, there is an n-fold rotational symmetry and an additional invariance under reflection. A case of particular interest is that the spline space itself reflects these symmetries. In other words, there is no reason why we should treat one of the n D-patches sharing an extraordinary point differently from the others or why we should give preference to a particular direction of rotation around the center. So the set of quasi control points should be invariant under the shift

$$S: \mathbf{C}_{11}^j, \mathbf{C}_{21}^j, \mathbf{C}_{12}^j \mapsto \mathbf{C}_{11}^{j+1}, \mathbf{C}_{21}^{j+1}, \mathbf{C}_{12}^{j+1} \tag{5.7}$$

and the reflection

$$R: \mathbf{C}_{11}^j, \mathbf{C}_{21}^j, \mathbf{C}_{12}^j \mapsto \mathbf{C}_{11}^{-j}, \mathbf{C}_{12}^{-j}, \mathbf{C}_{21}^{-j}. \tag{5.8}$$

This feature can be achieved readily by a special choice of constants.

Definition 5.4. A set of constants $\lambda = \{\alpha^j, \beta^j, \gamma^j, \delta^j, j = 1, ..., n\}$ is called *symmetric* if $\alpha^1 = \delta^1 = \cdots = \alpha^n = \delta^n = =: \alpha$ and $\beta^1 = \gamma^1 = \cdots = \beta^n = \gamma^n =: \beta$. The spline space $\mathscr{S}_\Omega^\Lambda$ is called *symmetric* if $\Lambda = \{\lambda_1, ..., \lambda_m\}$ consists of feasible symmetric sets.

Theorem 5.5. *A symmetric set of constants is feasible if* $\beta > 0$ *and*

$$\alpha = 1 - \beta \cos \varphi_n, \qquad \varphi_n := 2\pi/n. \tag{5.9}$$

Proof. Choosing $\alpha = 1 - \beta \cos \varphi_n$ implies that the eigenvalues of $\Phi := \Phi^1 = \cdots = \Phi^n$ are $\exp(\pm i\varphi_n)$. Hence, $\prod_{j=1}^{n} \Phi$ is the identity. ∎

From a purely analytical point of view the conditions specified in Theorem 5.5 are not necessary. In general, choosing $\alpha = 1 - \beta \cos(2\pi k/n)$, $k \in \{1, ..., n-1\} \backslash \{n/2\}$ yields eigenvalues $\exp(\pm 2\pi ki/n)$ and, hence, a feasible set. The cases $k = 0$ and, for n even, $k = n/2$ are leading to double eigenvalues and can be ruled out by inspection. However, one can show using discrete Fourier analysis that $k \notin \{1, n-1\}$ is leading to inadequately arranged quasi control points corresponding to surfaces with local self-intersections near the extraordinary point. In the sense of differential geometry such surfaces are not smooth and thus (5.9) turns out to be both necessary and sufficient.

Finally, let us discuss the refinement of $\mathcal{S}_\Omega^\Lambda$ induced by $F: \mathcal{S}_\Omega \mapsto \mathcal{S}_{\tilde{\Omega}}$. It turns out that the refined space $\mathcal{S}_{\tilde{\Omega}}^{\tilde{\Lambda}} := F(\mathcal{S}_\Omega^\Lambda)$ is linear, again. However, the process is not uniform in the way that the sets of constants Λ and $\tilde{\Lambda}$ are different.

THEOREM 5.6. *Consider the space* $\mathcal{S}_\Omega^\Lambda$ *with* $\Lambda := \{\lambda_1, ..., \lambda_m\}$ *and* $\lambda_\mu := \{\alpha_\mu^j, \beta_\mu^j, \gamma_\mu^j, \delta^j{}_\mu, j = 1, ..., n_\mu\}$, $\mu = 1, ..., m$. *Refinement by F as defined in the proof of Theorem* 4.1 *yields the linear space* $\mathcal{S}_{\tilde{\Omega}}^{\tilde{\Lambda}} := F(\mathcal{S}_\Omega^\Lambda) \subset \mathcal{S}_{\tilde{\Omega}}$, *where*

$$\begin{pmatrix} \tilde{\alpha}_\mu^j & \tilde{\beta}_\mu^j \\ \tilde{\gamma}_\mu^j & \tilde{\delta}_\mu^j \end{pmatrix} := \frac{1}{2} \begin{pmatrix} \alpha_\mu^j + 1 & \beta_\mu^j \\ \gamma_\mu^j & \delta_\mu^j + 1 \end{pmatrix}.$$

The symmetric case results in $\tilde{\beta}_\mu := \beta_\mu/2$ *and* $\tilde{\alpha}_\mu := (\alpha_\mu + 1)/2 = 1 - \tilde{\beta}_\mu \cos \varphi_n$.

Proof. The proof follows immediately from (4.5) and (4.6). ∎

6. CONTROL POINTS AND B-SPLINES

The features of the quasi control point construction are ambivalent. On one hand, their topological structure is a natural generalization of the familiar tensor product setup. On the other hand, the constraints imposed on them make them somewhat awkward to deal with. In particular, they will not be accepted by designers willing to model intuitively rather than to solve equations. One way out of this dilemma would be to identity the free parameters explicitly and to use them exclusively as control points. For the linear space $\mathcal{S}_\Omega^\Lambda$ this is quite simple. For instance, $\mathbf{C}_{11}^1, \mathbf{C}_{21}^1, \mathbf{C}_{12}^1$ can be chosen arbitrarily and then all other quasi control points will be determined uniquely as outlined in the preceding section. Thus, $3n$ quasi control points could be replaced by three proper control points assigned to one of the n D-patches sharing an extraordinary vertex. However, the fundamental drawback of this and all similar procedures is that the topology of the resulting control mesh does not match the patch structure uniformly. By assigning four control points to one D-patch and only one control point to each of the remaining ones the natural equivalence of patches would be violated inadmissibly. So, we propose a different approach, which is both satisfactory from a theoretical point of view and useful for applications. The idea is to start with arbitrarily located proper control points $\mathbf{c}_{11}^j$, $\mathbf{c}_{21}^j$, $\mathbf{c}_{12}^j$ which have exactly the same topological structure as the quasi control points but do not necessarily comply with the constraints. The next step is to project these points to the space of quasi control points and then one can proceed as described above. For instance, a convenient projection can be defined by selecting those quasi control points which minimize the distance to the given points with respect to some norm. This projection can be computed numerically or even analytically, if the Euclidean norm is used. However, we shall not elaborate on this problem in full generality but confine ourselves to a more detailed discussion of the linear symmetric case, which is of particular importance for applications.

The given proper control points $\mathbf{c}_{11}^j$, $\mathbf{c}_{21}^j$, $\mathbf{c}_{12}^j$ and the quasi control points $\mathbf{C}_{11}^j$, $\mathbf{C}_{21}^j$, $\mathbf{C}_{12}^j$, which have to be determined, are collected in vectors, e.g., $\mathbf{c}_{11} := [\mathbf{c}_{11}^0, ..., \mathbf{c}_{11}^{n-1}]^{\mathrm{T}}$ and analogously for all others. What we are looking for is an affine invariant projection,

$$P\colon \begin{pmatrix} \mathbf{c}_{11} \\ \mathbf{c}_{21} \\ \mathbf{c}_{12} \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{C}_{11} \\ \mathbf{C}_{21} \\ \mathbf{C}_{12} \end{pmatrix} = \begin{pmatrix} P_1 & P_2 & P_3 \\ P_4 & P_5 & P_6 \\ P_7 & P_8 & P_9 \end{pmatrix} \begin{pmatrix} \mathbf{c}_{11} \\ \mathbf{c}_{21} \\ \mathbf{c}_{12} \end{pmatrix}, \tag{6.1}$$

mapping proper control points to the space of quasi control points. For symmetry reasons, it is required that P commutes with both the shift S and the reflection R, that is

$$SP = PS, \qquad RP = PR,$$

where with a slight abuse of notation S and R are $(3n \times 3n)$-matrices representing (5.7) and (5.8). $SP = PS$ implies that all $n \times n$-matrices P_ν, $\nu = 1, ..., 9$, are *cyclic*, that is there exist vectors $p_\nu := [p_\nu^0, ..., p_\nu^{n-1}]^{\mathrm{T}}$ such that $P_\nu^{jk} = p_\nu^{j-k}$, $j, k = 0, ..., n-1$. Further, $RP = PR$ yields

$$p_1^j = p_1^{-j}, \qquad p_2^j = p_3^{-j}, \qquad p_4^j = p_7^{-j}, \qquad p_5^j = p_9^{-j}, \qquad p_6^j = p_8^{-j} \tag{6.2}$$

by inspection. The analysis of cyclic systems can be simplified considerably using the discrete Fourier transform $p \mapsto \hat{p}$ defined by

$$\hat{p}^k := \sum_{j=0}^{n-1} \omega_n^{-jk} p^j, \qquad p^j = \frac{1}{n} \sum_{k=0}^{n-1} \omega_n^{jk} \hat{p}^k, \qquad \omega_n := \exp(i\varphi_n); \tag{6.3}$$

see [13] for details. Applying it to the space of control points splits (6.1) into n decoupled 3×3-systems,

$$\hat{P}^k: \begin{pmatrix} \hat{\mathbf{c}}_{11}^k \\ \hat{\mathbf{c}}_{21}^k \\ \hat{\mathbf{c}}_{12}^k \end{pmatrix} \mapsto \begin{pmatrix} \hat{\mathbf{C}}_{11}^k \\ \hat{\mathbf{C}}_{21}^k \\ \hat{\mathbf{C}}_{12}^k \end{pmatrix} = \begin{pmatrix} \hat{p}_1^k & \hat{p}_2^k & \hat{p}_3^k \\ \hat{p}_4^k & \hat{p}_5^k & \hat{p}_6^k \\ \hat{p}_7^k & \hat{p}_8^k & \hat{p}_9^k \end{pmatrix} \begin{pmatrix} \hat{\mathbf{c}}_{11}^k \\ \hat{\mathbf{c}}_{21}^k \\ \hat{\mathbf{c}}_{12}^k \end{pmatrix} \tag{6.4}$$

and the symmetry conditions (6.2) imply

$$\hat{p}_1^k = \overline{\hat{p}_1^k}, \qquad \hat{p}_2^k = \overline{\hat{p}_3^k}, \qquad \hat{p}_4^k = \overline{\hat{p}_7^k}, \qquad \hat{p}_5^k = \overline{\hat{p}_9^k}, \qquad \hat{p}_6^k = \overline{\hat{p}_8^k}. \tag{6.5}$$

By means of (6.3), the constraints on quasi control points as specified in Definition 5.2 are transformed to

$$\begin{pmatrix} 1-\omega_n^k & 0 & 0 \\ 1-\alpha-\beta & (\alpha+\beta\omega_n^k)/2-1 & (\alpha\omega_n^{-1}+\beta)/2 \\ 1-\alpha-\beta & (\alpha\omega_n^k+\beta)/2 & (\alpha+\beta\omega_n^{-1})/2-1 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{C}}_{11}^k \\ \hat{\mathbf{C}}_{21}^k \\ \hat{\mathbf{C}}_{12}^k \end{pmatrix} = \mathbf{0}. \tag{6.6}$$

Denoting the above matrix by $\hat{Q}^k$, the projection P has to satisfy $\hat{Q}^k \hat{P}^k = 0$. With $\alpha = 1 - \beta \cos \varphi_n$ we find

$$\det \hat{Q}^k = \beta(1-\omega_n^k)(\cos(k\varphi_n) - \cos(\varphi_n)) \tag{6.7}$$

and the kernel of $\hat{Q}^k$ is nontrivial if and only if $k \in \{0, 1, n-1\}$; thus

$$\hat{P}^2 = \cdots = \hat{P}^{n-2} = 0. \tag{6.8}$$

For $k=0$ we obtain $\hat{q}^0 := \ker \hat{Q}^0 = [1, 1, 1]^{\mathrm{T}}$ and $\hat{P}^0 := \hat{q}^0[a_0^0, a_1^0, a_2^0]$. The symmetry conditions (6.5), the affine invariance of P, and the fact that $\hat{P}^0$ has real entries imply

$$a_1^0 = a_2^0 = (1-a_0^0)/2, \qquad a_0^0 \in \mathbb{R}. \tag{6.9}$$

For $k=1$ we obtain $\hat{q}^1 := \ker \hat{Q}^1 = [0, \exp(i\psi), \exp(-i\psi)]^{\mathrm{T}}$, $\psi := \arg((1+i\beta \sin \varphi_n)\, \omega_n^{-1/2})$, and $\hat{P}^1 := \hat{q}^1[a_0^1, a_1^1, a_2^1]$. The symmetry conditions (6.5) imply that $a_0^1 \in \mathbb{R}$ and

$$a_1^1 = \overline{a_2^1} =: r \exp(i\tau), \qquad (r, \tau) \in \mathbb{R}_0^+ \times [0, 2\pi). \tag{6.10}$$

For $k=n-1$ we obtain $\hat{q}^{n-1} := \ker \hat{Q}^{n-1} = \overline{\hat{q}^1}$ and setting $\hat{P}^{n-1} = \overline{\hat{P}^1}$ ensures that P is a real matrix. So the projection P is characterized by four real-valued parameters a_0^0, a_0^1, r, τ and the vectors $p_1, ..., p_9$ given by

$$\begin{aligned}
p_1^j &= a_0^0/n \\
p_2^j &= (1-a_0^0)/2n \\
p_3^j &= (1-a_0^0)/2n \\
p_4^j &= (a_0^0 + 2a_0^1 \cos(j\varphi_n))/n \\
p_5^j &= ((1-a_0^0) + 4r\cos(\psi+\tau+j\varphi_n))/2n \\
p_6^j &= ((1-a_0^0) + 4r\cos(\psi-\tau+j\varphi_n))/2n \\
p_7^j &= (a_0^0 + 2a_0^1 \cos(j\varphi_n))/n \\
p_8^j &= ((1-a_0^0) + 4r\cos(\psi-\tau-j\varphi_n))/2n \\
p_9^j &= ((1-a_0^0) + 4r\cos(\psi+\tau-j\varphi_n))/2n
\end{aligned} \tag{6.11}$$

for $j=0, ..., n-1$. How do we choose the parameters? A natural criterion is to minimize the distance

$$\sum_{j=0}^{n-1} (\|\hat{\mathbf{C}}_{11}^j - \hat{\mathbf{c}}_{11}^j\|^2 + \|\hat{\mathbf{C}}_{21}^j - \hat{\mathbf{c}}_{21}^j\|^2 + \|\hat{\mathbf{C}}_{12}^j - \hat{\mathbf{c}}_{12}^j\|^2) \to \min. \tag{6.12}$$

Some elementary calculus yields $a_0^0 = \frac{1}{3}$, $a_0^1 = 0$, $r = \frac{1}{2}$, $\tau = -\psi$, and

$$\begin{aligned}
p_1^j = p_2^j = p_3^j = p_4^j = p_7^j &= 1/3n \\
p_5^j = p_9^j &= (1 + 3\cos(j\varphi_n))/3n \\
p_6^j &= (1 + 3\cos(2\psi + j\varphi_n))/3n \\
p_8^j &= (1 + 3\cos(2\psi - j\varphi_n))/3n.
\end{aligned} \tag{6.13}$$

Some of the coefficients are negative and, consequently, the quasi control points, and because of them the spline surface will not necessarily lie in the convex hull of the proper control points. The projection complies with the convex hull property if the parameters satisfy $0 \leqslant a_0^0 \leqslant 1$, $|a_0^1| \leqslant a_0^0/2$, $r \leqslant (1-a_0^0)/4$. If this is desired a reasonable choice is $a_0^0 = a_0^1 = 0$, $r = \frac{1}{4}$, $\tau = -\psi$, yielding

$$\begin{aligned}
p_1^j = p_4^j = p_7^j &= 0 \\
p_2^j = p_3^j &= 1/2n \\
p_5^j = p_9^j &= (1 + \cos(j\varphi_n))/2n \\
p_6^j &= (1 + \cos(2\psi + j\varphi_n))/2n \\
p_8^j &= (1 + \cos(2\psi - j\varphi_n))/2n.
\end{aligned} \tag{6.14}$$

As mentioned above, providing unrestricted control points is of particular importance for design purposes. As usual, the designer specifies proper control points and the result is a smooth spline surface. All intermediate steps and, in particular, the projection to the space of quasi control points, can be hidden in a black box. Actually, the same process turns out to be equally useful from a more theoretical point of view. Recall the generation of a point $\mathbf{x}(p)$, $p \in \Omega = \omega \times I$ on the spline surface $\mathbf{x}$, corresponding to a set of control points $\mathbf{c} = \{\mathbf{c}_i, i = 1, ..., 4\#I\}$. First, quasi control points $\mathbf{C}$ are computed by applying the projection P to the proper control points $\mathbf{c}$. Then the quasi control points are completed to a set of Bézier points by averaging as described in the third section, and finally $\mathbf{x}(p)$ is obtained by evaluating the appropriate Bézier patch. Combining all three steps yields a single linear and affine invariant map acting on control points according to

$$\mathscr{B}_\Omega : \mathbf{c} \mapsto \mathbf{x} \in \mathscr{S}^A_\Omega, \qquad \mathbf{x}(p) = \sum_{i=1}^{4\#I} B_i(p)\, \mathbf{c}_i. \tag{6.15}$$

The functions $B_i : \Omega \mapsto \mathbb{R}$ are *B-splines* in the following sense: They are real-valued, piecewise polynomial, compactly supported, form a partition of unity, and coincide with ordinary B-splines on the regular parts of the domain. Partition of unity is an immediate consequence of the fact that the rows of P sum up to one by construction. Nonnegativity can be achieved using particular projections P, for instance, (6.14). Further, they provide built-in smoothness by generating smooth surfaces when combined linearly with arbitrary spatial control points. They span the complete space $\mathscr{S}^A_\Omega$ but do not form a basis since they are linearly dependent as a consequence of the rank deficiency of the projection matrix P.

CONCLUSION

Degenerate surface patches are suitable for modeling smooth spline surfaces. The major benefit of this approach is the capability of modeling a refineable spline space of arbitrary topological genus provided with a uniform set of parametric smoothness conditions. The existence of a nontrivial linear subspace spanned by a family of real-valued compactly supported B-spline functions favors various applications in both computer aided geometric design and approximation theory. Disadvantages of the methods are that it requires polynomials of bi-degree three for modeling C^1-surfaces and that the main curvatures diverge near singularly parametrized surface points. Recent results show that the approach using singular parametrization can be generalized to surfaces with continuous curvature [4, 5]. However, the conditions derived there are highly nonlinear and do not admit a geometrically meaningful interpretation. A slightly different approach providing refineable spline spaces of arbitrary smoothness order is proposed in [20].

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